

# Criterion for SLOCC Equivalence of Multipartite Quantum States

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We study the stochastic local operation and classical communication (SLOCC) equivalence for arbitrary dimensional multipartite quantum states. For multipartite pure states, we present a necessary and sufficient criterion in terms of their coefficient matrices. This condition can be used to classify some SLOCC equivalent quantum states with coefficient matrices having the same rank. For multipartite mixed state, we provide a necessary and sufficient condition by means of the realignment of matrix. Some detailed examples are given to identify the SLOCC equivalence of multipartite quantum states.

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## I. INTRODUCTION

Quantum entanglement is not only a prime feature in quantum mechanics but also an important resource in quantum information processes [1, 2]. It can be used in quantum teleportation [3, 4], superdense coding [5, 6], quantum computation [7–10], quantum key distribution [11, 12] and etc. Therefore, it is important to understand what kind of entanglement a given quantum state has. One approach to classify entanglement is by means of Statistic local operations and classical communications (SLOCC) [13]. Entanglement in bipartite pure states has been well understood, while many questions are still open for the mixed states and multipartite states.

It has been shown that two pure states  $|\varphi\rangle$  and  $|\psi\rangle$  in  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_K$ ,  $\dim \mathcal{H}_i = n_i$ ,  $i = 1, 2, \dots, K$ , are SLOCC equivalent if and only if they can be converted into each other with the tensor products of invertible local operators (ILOs)

$$|\varphi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_K |\psi\rangle. \quad (1)$$

Correspondingly, two mixed states  $\rho$  and  $\rho'$  belong to the same class under SLOCC if and only if they are converted by ILOs with nonzero determinant, that is,

$$\rho' = (A_1 \otimes A_2 \otimes \cdots \otimes A_K) \rho (A_1 \otimes A_2 \otimes \cdots \otimes A_K)^\dagger, \quad (2)$$

where  $A_i$  is ILO in  $GL(n_i, \mathbb{C})$  for each  $i$  [14]. Many researches have been conducted on entanglement classification under SLOCC since the beginning of this century [14–24]. In three-qubit system, all pure states are classified into six types [14]. This classification can be extended to three-qubit mixed states [15]. Even though, it is still a very difficult problem to find a SLOCC class of a given three-qubit mixed state except for a few rare case. For instance, a complete SLOCC classification for the set of the GHZ-symmetric states was reported in Ref. [16]. In four-qubit case, all pure states are classified into nine SLOCC inequivalent families using group theory [17]. For  $n$ -qubit system, Ref. [22] uses the ranks of the coefficient matrices to study SLOCC classification for pure state. Then Ref. [23] generalizes Li's approach to  $n$ -qudit pure state. Recently, Ref. [24] shows that almost all SLOCC equivalent classes can be distinguished by ratios of homogeneous SL-invariant polynomials of the same degree. Theoretically, their technique can be applied to any number of qudits in all dimensions. But, it is still a significant challenge to find a general scheme that is able to completely identify the different entanglement classes and determine the transformation matrices connecting two equivalent states under SLOCC for multipartite mixed states. In Ref. [25], we have constructed a nontrivial set of invariants for any multipartite mixed states under SLOCC.

In this paper we present a general scheme for the SLOCC equivalence of arbitrary dimensional multipartite quantum pure or mixed states in terms of matrix realignment [26, 27]. In Sec. II, we recall some basic results, then we give the criterion for how to judge a block invertible matrix can be decomposed as the tensor products of invertible matrices. In Sec. III, we give a necessary and sufficient criterion for the SLOCC equivalence of multipartite pure states. For the multipartite mixed states, we propose a similar criterion based on the density matrix itself in Sec. IV. These criteria are shown to be still operational for general states, and we also give the explicit forms of the connecting matrix for two SLOCC equivalent states in specific examples. At last, we give the conclusions and remarks.

## II. TENSOR PRODUCTS DECOMPOSITION FOR BLOCK INVERTIBLE MATRIX

First we introduce the definitions for realignment of matrix [26, 27].

**Definition 1:** For any  $M \times N$  matrix  $A$  with entries  $a_{ij}$ ,  $\text{vec}(A)$  is defined by

$$\text{vec}(A) \equiv [a_{11}, \dots, a_{M1}, a_{12}, \dots, a_{M2}, \dots, a_{1N}, \dots, a_{MN}]^T,$$

where  $T$  denotes transposition.

**Definition 2:** Let  $Z$  be an  $M \times M$  block matrix with each block of size  $N \times N$ , the realigned matrix  $R(Z)$  is defined by  $R(Z) \equiv [\text{vec}(Z_{11}), \dots, \text{vec}(Z_{M1}), \dots, \text{vec}(Z_{1M}), \dots, \text{vec}(Z_{MM})]^T$ .

Based on the definitions of realignment, Ref. [28] shows a necessary and sufficient condition for the tensor products decomposition of invertible matrices for a matrix.

**Lemma 1.** An  $MN \times MN$  invertible matrix  $A$  is expressed as the tensor product of an  $M \times M$  invertible matrix  $A_1$  and an  $N \times N$  invertible matrix  $A_2$ , i.e.,  $A = A_1 \otimes A_2$  if and only if  $\text{rank } R(A) = 1$ .

For any  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  matrix  $A$ , we denote  $A_{i|\hat{i}}$  the  $N_i \times N_i$  block matrix with each block of size  $N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_K \times N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_K$ . Namely, we view  $A$  as a bipartite partitioned matrix  $A_{i|\hat{i}}$  with partitions  $H_i$  and  $H_1 \otimes H_2 \cdots H_{i-1} \otimes H_{i+1} \cdots H_K$ . Accordingly, we have the realigned matrix  $R(A_{i|\hat{i}})$ .

**Theorem 1.** Let  $A$  be an  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  invertible matrix, there exist  $N_i \times N_i$  invertible matrices  $a_i$ ,  $i = 1, 2, \dots, K$ , such that  $A = a_1 \otimes a_2 \otimes \cdots \otimes a_K$  if and only if the  $\text{rank}(R(A_{i|\hat{i}})) = 1$  for all  $i$ .

*Proof.* First, if there exist  $N_i \times N_i$  invertible matrices  $a_i$ ,  $i = 1, 2, \dots, K$ , such that  $A = a_1 \otimes a_2 \otimes \cdots \otimes a_K$ , by viewing  $A$  in bipartite partition and using Lemma 1, one has directly that  $\text{rank}(R(A_{i|\hat{i}})) = 1$  for all  $i$ .

On the other hand, if  $\text{rank}(R(A_{i|\hat{i}})) = 1$ , for any given  $i$ , we prove the conclusion by induction. First, for  $n = 3$ , from Lemma 1, we have  $A = a_1 \otimes a_{23} = a_2 \otimes a_{13}$ . Multiplying  $a_1^{-1}$  for the first subsystem from the left, it has  $(a_1^{-1} \otimes I_2 \otimes I_3)A = I_1 \otimes a_{23} = a_2 \otimes ((a_1^{-1} \otimes I_3)a_{13})$ . By tracing out the first subsystem, we get  $N_1 a_{23} = a_2 \otimes \text{Tr}_1((a_1^{-1} \otimes I_3)a_{13})$ , i.e.,  $a_{23} = a_2 \otimes a'_3$  with invertible matrix  $a'_3 = \text{Tr}_1((a_1^{-1} \otimes I_3)a_{13})/N_1$ . Assume that the conclusion is also true for  $K - 1$ , then for  $K$ , from Lemma 1, we have  $A = a_1 \otimes a_{\hat{1}} = a_2 \otimes a_{\hat{2}} = \cdots = a_K \otimes a_{\hat{K}}$ , where  $a_i$  is an  $N_i \times N_i$  invertible matrix and  $a_{\hat{i}}$  is an  $N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_K \times N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_K$  invertible matrix,  $i = 1, 2, \dots, K$ . Hence  $(I_1 \otimes \cdots \otimes I_{K-1} \otimes a_K^{-1})A = (I_1 \otimes \cdots \otimes I_{K-1} \otimes a_K^{-1})(a_1 \otimes a_{\hat{1}}) = \cdots = (I_1 \otimes \cdots \otimes I_{K-1} \otimes a_K^{-1})(a_{\hat{K}} \otimes a_K)$ . Tracing out the last subsystem we get  $a_1 \otimes \text{Tr}_K(I_2 \otimes \cdots \otimes I_{N_{K-1}} \otimes a_K^{-1} a_{\hat{1}}) = \cdots = \text{Tr}_K((I_1 \otimes \cdots \otimes I_{K-2} \otimes a_K^{-1}) \otimes (a_{K-1})) = N_K a_{\hat{K}}$ . Based on the assumption, we know  $a_{\hat{K}}$  can be written as the tensor products of local invertible operators. Therefore,  $A$  also can be written as the tensor products of local invertible operators, which completes the proof.  $\square$

## III. CRITERION FOR MULTIPARTITE PURE STATES

First, we recall the notations of coefficient matrices of pure state [22, 23]. Let  $\{|i_1\rangle\}_{i_1=0}^{n_1-1}$ ,  $\{|i_2\rangle\}_{i_2=0}^{n_2-1}$ ,  $\dots$ ,  $\{|i_K\rangle\}_{i_K=0}^{n_K-1}$  be orthonormal basis of  $K$  Hilbert spaces  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ ,  $\dots$ ,  $\mathcal{H}_K$ . For any  $K$  partite pure state  $|\psi\rangle = \sum_{i_1, i_2, \dots, i_K=0}^{n_1-1, n_2-1, \dots, n_K-1} a_{i_1 i_2, \dots, i_K} |i_1 i_2, \dots, i_K\rangle$ ,  $\sum_{i_1, i_2, \dots, i_K=0}^{n_1-1, n_2-1, \dots, n_K-1} |a_{i_1 i_2, \dots, i_K}|^2 = 1$ , we associate an  $m \times n$  coefficient matrix  $M(|\psi\rangle)$  to it,  $m = n_1 n_2 \cdots n_t$ ,  $n = n_{t+1} \cdots n_K$ ,  $t = \lfloor \frac{K}{2} \rfloor$ .

For example, for three qubit pure state  $|\psi\rangle = \sum_{i_1, i_2, i_3=0}^1 a_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle$ , we have the  $2 \times 4$  coefficient matrices:

$$M(|\psi\rangle) = \begin{pmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \end{pmatrix}.$$

For four qubit pure state  $|\psi\rangle = \sum_{s_1, s_2, s_3, s_4=0}^1 a_{s_1 s_2 s_3 s_4} |s_1 s_2 s_3 s_4\rangle$ , there is  $4 \times 4$  coefficient matrices, that is:

$$M(|\psi\rangle) = \begin{pmatrix} a_{0000} & a_{0001} & a_{0010} & a_{0011} \\ a_{0100} & a_{0101} & a_{0110} & a_{0111} \\ a_{1000} & a_{1001} & a_{1010} & a_{1011} \\ a_{1100} & a_{1101} & a_{1110} & a_{1111} \end{pmatrix}.$$

Using the rank of coefficient matrix  $M(|\psi\rangle)$ , Refs. [22, 23] classified multipartite pure states into different families. If the coefficient matrices of two pure states have different ranks, then these two pure states are not SLOCC equivalent.

While the converse does not hold true, i.e. if the coefficient matrices have the same rank, then corresponding pure states are not necessarily SLOCC equivalent. Here we answer this question further when two states with the same rank of the coefficient matrices are equivalent under SLOCC.

**Theorem 2.** *For two  $K$ -partite pure states  $|\phi\rangle$  and  $|\psi\rangle$ , they are SLOCC equivalent if and only if for one pair of coefficient matrices  $M(|\phi\rangle)$  and  $M(|\psi\rangle)$ , there are  $m \times m$  unitary matrices  $X_1, X_2$ , invertible diagonal matrix  $B_1$ , and  $n \times n$  unitary matrices  $Y_1, Y_2$ , invertible diagonal matrix  $B_2$ , such that*

$$M(|\phi\rangle) = X_1 B_1 X_2^\dagger M(|\psi\rangle) Y_2^\dagger B_2 Y_1, \quad (3)$$

and

$$\text{rank}[R((X_1 B_1 X_2^\dagger)_{i|\hat{i}})] = 1 \quad (4)$$

and

$$\text{rank}[R((Y_2^\dagger B_2 Y_1)_{j|\hat{j}})] = 1, \quad (5)$$

$i = 1, 2, \dots, t, j = t + 1, \dots, K$ .

*Proof.* First, suppose  $|\phi\rangle$  and  $|\psi\rangle$  are SLOCC equivalent, i.e. there exist invertible matrices  $C_1, C_2, \dots, C_K$  such that  $|\phi\rangle = (C_1 \otimes C_2 \otimes \dots \otimes C_K)|\psi\rangle$ . In matrix form,

$$M(|\phi\rangle) = (C_1 \otimes C_2 \otimes \dots \otimes C_t) M(|\psi\rangle) (C_{t+1} \otimes \dots \otimes C_K)^T. \quad (6)$$

For invertible matrices  $C_1 \otimes C_2 \otimes \dots \otimes C_t$  and  $(C_{t+1} \otimes \dots \otimes C_K)^T$ , by the singular value decomposition of a matrix, there exist  $m \times m$  unitary matrices  $X_1, X_2$ , invertible diagonal matrix  $B_1$ , and  $n \times n$  unitary matrices  $Y_1, Y_2$ , invertible diagonal matrix  $B_2$  such that:

$$\begin{aligned} C_1 \otimes C_2 \otimes \dots \otimes C_t &= X_1 B_1 X_2^\dagger, \\ (C_{t+1} \otimes \dots \otimes C_K)^T &= Y_1 B_2 Y_2^\dagger. \end{aligned}$$

Inserting these decompositions into Eq. (6), one gets easily Eq. (3). By Lemma 1, we can get Eqs. (4) and (5),  $i = 1, 2, \dots, t, j = t + 1, \dots, K$ .

On the other hand, suppose there exist one pair of coefficient matrices  $M(|\phi\rangle)$  and  $M(|\psi\rangle)$  of  $|\phi\rangle$  and  $|\psi\rangle$  satisfying the conditions mentioned in the Theorem. By Lemma 1, we know there are invertible matrices  $C_1, C_2, \dots, C_K$  such that Eq. (6) holds true. Therefore  $|\phi\rangle = (C_1 \otimes C_2 \otimes \dots \otimes C_K)|\psi\rangle$ , i.e.  $|\phi\rangle$  and  $|\psi\rangle$  are SLOCC equivalent.  $\square$

Let us now take a closer look at equations in Theorem 2. Eq. (6) means if two pure states are SLOCC equivalent, then their coefficient matrices have the same rank. Eqs. (4) and (5) means if two pure states are SLOCC equivalent, then their coefficient matrices are connected by tensor products of invertible matrices. So if the coefficient matrices have the same rank, then one needs to verify Eqs. (4) and (5) to check whether two pure states are SLOCC equivalent or not.

Operationally, for two pure states  $|\phi\rangle$  and  $|\psi\rangle$ , we first choose one kind of coefficient matrices  $M(|\phi\rangle)$  and  $M(|\psi\rangle)$ . If  $M(|\phi\rangle)$  and  $M(|\psi\rangle)$  have different ranks, then  $|\phi\rangle$  and  $|\psi\rangle$  are not SLOCC equivalent. If  $M(|\phi\rangle)$  and  $M(|\psi\rangle)$  have the same rank, then by the singular value decomposition, there are  $m \times m$  unitary matrices  $X_1, X_2$ , diagonal matrix  $\Lambda_1$ , and  $n \times n$  unitary matrices  $Y_1, Y_2$ , diagonal matrix  $\Lambda_2$  such that:

$$M(|\phi\rangle) = X_1 \Lambda_1 Y_1 \quad (7)$$

and

$$M(|\psi\rangle) = X_2 \Lambda_2 Y_2, \quad (8)$$

where  $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0)$ ;  $\Lambda_2 = \text{diag}(\mu_1, \mu_2, \dots, \mu_r, 0, \dots, 0)$ ,  $\lambda_i$  and  $\mu_i$  are nonzero real numbers. Let  $m \times m$  invertible matrix  $B_1 = \text{diag}(\sqrt{\frac{\lambda_1}{\mu_1}}, \sqrt{\frac{\lambda_2}{\mu_2}}, \dots, \sqrt{\frac{\lambda_r}{\mu_r}}, 1, \dots, 1)$  and  $n \times n$  invertible matrix  $B_2 = \text{diag}(\sqrt{\frac{\lambda_1}{\mu_1}}, \sqrt{\frac{\lambda_2}{\mu_2}}, \dots, \sqrt{\frac{\lambda_r}{\mu_r}}, 1, \dots, 1)$ , then one has  $\Lambda_1 = B_1 \Lambda_2 B_2$  and  $M(|\phi\rangle) = X_1 B_1 X_2^\dagger M(|\psi\rangle) Y_2^\dagger B_2 Y_1$ . Next one needs to calculate the ranks for the realignment of  $X_1 B_1 X_2^\dagger$  and  $Y_2^\dagger B_2 Y_1$  under all partitions to see whether it is one or not.

For bipartite pure state  $|\phi\rangle = \sum_{i_1, i_2=0}^{n_1-1, n_2-1} a_{i_1 i_2} |i_1 i_2\rangle$ , there is only one way to express its coefficients in matrix form,  $M(|\phi\rangle) = (a_{i_1 i_2})$ . Therefore, two bipartite pure states  $|\phi\rangle$  and  $|\psi\rangle$  are SLOCC equivalence if and only if there exist invertible matrices  $C_1, C_2$  such that

$$M(|\phi\rangle) = C_1 M(|\psi\rangle) C_2^T.$$

Or equivalently, two bipartite pure states  $|\phi\rangle$  and  $|\psi\rangle$  are SLOCC equivalence if and only if their coefficient matrices have the same rank.

#### IV. CRITERION FOR MULTIPARTITE MIXED STATES

**Theorem 3.** *For two multipartite mixed quantum states  $\rho_1$  and  $\rho_2$ , they are SLOCC equivalent if and only if there exist  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  unitary matrices  $X$  and  $Y$ , real diagonal invertible matrix  $B$ , such that*

$$\rho_1 = X B Y^\dagger \rho_2 Y B X^\dagger, \quad (9)$$

and

$$\text{rank}(R(X B Y^\dagger)_{i|\hat{i}}) = 1, \quad (10)$$

for  $i = 1, 2, \dots, K$ .

*Proof.* If  $\rho_1$  and  $\rho_2$  are SLOCC equivalent, then there exist invertible matrices  $a_1, a_2, \dots, a_K$  such that  $(a_1 \otimes a_2 \otimes \cdots \otimes a_K) \rho_1 (a_1 \otimes a_2 \otimes \cdots \otimes a_K)^\dagger = \rho_2$ . For matrix  $a_1 \otimes a_2 \otimes \cdots \otimes a_K$ , by singular value decomposition, there exist  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  unitary matrices  $X, Y$ , real diagonal invertible matrix  $B$ , such that  $a_1 \otimes a_2 \otimes \cdots \otimes a_K = X B Y^\dagger$ . Then  $R(X B Y^\dagger) = R(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$ . From Lemma 1,  $\text{rank}(R(X B Y^\dagger)_{i|\hat{i}}) = 1$ , for  $i = 1, 2, \dots, K$ .

On the other hand, if there exist  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  unitary matrices  $X$  and  $Y$ , real diagonal invertible matrix  $B$ , such that Eq. (9) holds true and  $\text{rank}(R(X B Y^\dagger)_{i|\hat{i}}) = 1$  for  $i = 1, 2, \dots, K$ , then by Lemma 1, there exist invertible matrices  $a_1, a_2, \dots, a_K$  such that  $X B Y^\dagger = a_1 \otimes a_2 \otimes \cdots \otimes a_n$ . Inserting this equation into Eq. (9), one gets  $(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^\dagger \rho_1 (a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \rho_2$ , which ends the proof.  $\square$

Eq. (9) means if two mixed states are SLOCC equivalent, then they have the same rank. Eq. (10) means if two mixed states are SLOCC equivalent, then they are connected by the tensor products of invertible matrices. Now we show how to verify Theorem 3 explicitly. For two mixed states  $\rho_1$  and  $\rho_2$ , if they have different ranks, then they are not SLOCC equivalent. Or else, if  $\rho_1$  and  $\rho_2$  have the same rank, then we first study their spectra decompositions,

$$\rho_1 = X \Lambda_1 X^\dagger, \quad \rho_2 = Y \Lambda_2 Y^\dagger, \quad (11)$$

where  $X = [x_1, x_2, \dots, x_{N_1 N_2 \cdots N_K}]$ ,  $Y = [y_1, y_2, \dots, y_{N_1 N_2 \cdots N_K}]$ ,  $\{x_i\}$  and  $\{y_i\}$  are the normalized eigenvectors of states  $\rho_1$  and  $\rho_2$ .  $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0)$ ;  $\Lambda_2 = \text{diag}(\mu_1, \mu_2, \dots, \mu_r, 0, \dots, 0)$ ,  $\lambda_i$  and  $\mu_i$  are nonzero real numbers. For diagonal matrices  $\Lambda_1$  and  $\Lambda_2$ , there exists  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  invertible matrix

$$B = \text{diag}\left(\sqrt{\frac{\lambda_1}{\mu_1}}, \sqrt{\frac{\lambda_2}{\mu_2}}, \dots, \sqrt{\frac{\lambda_r}{\mu_r}}, s, \dots, t\right) \quad (12)$$

such that

$$\Lambda_1 = B \Lambda_2 B,$$

where  $s, \dots, t$  are arbitrary nonzero numbers. Therefore, there exist  $N_1 N_2 \cdots N_K \times N_1 N_2 \cdots N_K$  unitary matrices  $X$  and  $Y$ , real diagonal invertible matrix  $B$ , such that Eq. (9) holds true. Next we need to verify the rank of realignment of  $X B Y^\dagger$  to see whether  $\rho_1$  and  $\rho_2$  are SLOCC equivalent or not.

Example 1. First, we consider two-qubit Bell-diagonal states in two-qubit system [29, 30]:

$$\rho_1 = \sum_{i=1}^4 \lambda_i |\psi_i\rangle \langle \psi_i|, \quad \lambda_i \geq 0, \quad \sum_{i=1}^4 \lambda_i = 1, \quad i = 1, 2, 3, 4;$$

$$\rho_2 = \sum_{i=1}^4 \mu_i |\psi_i\rangle\langle\psi_i|, \quad \mu_i \geq 0, \quad \sum_{i=1}^4 \mu_i = 1, \quad i = 1, 2, 3, 4;$$

with  $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ,  $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ ,  $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ ,  $|\psi_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ . By

spectra decomposition, we have  $X = Y = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ ;  $\Lambda_1 = \text{diag}(2\lambda_1, 2\lambda_2, 2\lambda_4, 2(1 - \lambda_1 - \lambda_2 - \lambda_4))$ ;

$\Lambda_2 = \text{diag}(2\mu_1, 2\mu_2, 2\mu_4, 2(1 - \mu_1 - \mu_2 - \mu_4))$ . For simplicity, we consider only the non-degenerate case, which means  $\Lambda_1$  and  $\Lambda_2$  are nonsingular. Let  $B = \text{diag}(\sqrt{\frac{\lambda_1}{\mu_1}}, \sqrt{\frac{\lambda_2}{\mu_2}}, \sqrt{\frac{\lambda_4}{\mu_4}}, \sqrt{\frac{\lambda_1 + \lambda_2 + \lambda_4 - 1}{\mu_1 + \mu_2 + \mu_4 - 1}})$ , then  $\rho_1$  and  $\rho_2$  satisfy Eq. (9). Next we need to study the rank of realignment matrix  $XY^\dagger = B$ . We find if

$$\sqrt{\frac{\lambda_1}{\mu_1}} : \sqrt{\frac{\lambda_4}{\mu_4}} = \sqrt{\frac{\lambda_2}{\mu_2}} : \sqrt{\frac{\lambda_1 + \lambda_2 + \lambda_4 - 1}{\mu_1 + \mu_2 + \mu_4 - 1}},$$

then  $\text{rank}(R(XBY^\dagger)) = 1$ . In this case,  $\rho_1$  and  $\rho_2$  are SLOCC equivalent.

Example 2. Now we consider two mixed states in  $2 \otimes 2 \otimes 2$  system,

$$\rho_1 = \frac{1}{K} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\rho_2 = \frac{1}{M} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the normalization factors  $K = 2 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ .  $M = 2 + \alpha + \beta + \gamma + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ . First we study the spectra decompositions of  $\rho_1$  and  $\rho_2$ . Here as in Eq. (11),  $\Lambda_1 = \frac{1}{K} \text{diag}(\frac{1}{c}, \frac{1}{b}, \frac{1}{a}, 2, a, b, c, 0)$  and  $\Lambda_2 = \frac{1}{M} \text{diag}(\frac{1}{\gamma}, \frac{1}{\beta}, \frac{1}{\alpha}, 2, \alpha, \beta, \gamma, 0)$ . To simplify the problem, suppose  $a, b, c, \alpha, \beta, \gamma$  take different values unequal to 0, 1,  $\frac{1}{2}$ , 2. Then we can easily get

$$X = Y = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Let  $B = \text{diag}(\sqrt{\frac{M\gamma}{Kc}}, \sqrt{\frac{M\beta}{Kb}}, \sqrt{\frac{M\alpha}{Ka}}, \sqrt{\frac{M}{K}}, \sqrt{\frac{Ma}{K\alpha}}, \sqrt{\frac{Mb}{K\beta}}, \sqrt{\frac{Mc}{K\gamma}}, C)$  with  $C$  an arbitrary nonzero number. Then  $XY^\dagger = B$ . Now we calculate the rank of the realignment of  $XY^\dagger$ . If the coefficients of  $\rho_1$  and  $\rho_2$  satisfies the following two condition,

$$(1) \sqrt{\frac{\gamma}{c}} : \sqrt{\frac{\alpha}{a}} = \sqrt{\frac{\beta}{b}} : \sqrt{\frac{c}{\beta}} = \sqrt{\frac{\alpha}{a}} : \sqrt{\frac{c}{\gamma}} = 1 : C,$$

$$(2)\sqrt{\frac{\alpha}{c}} : \sqrt{\frac{\beta}{b}} = \sqrt{\frac{\alpha}{a}} : 1$$

then  $\text{rank}(R(XBY^\dagger)_{i|\hat{i}}) = 1$  for  $i = 1, 2, 3$ . In this case,  $\rho_1$  and  $\rho_2$  are SLOCC equivalent. For instance, when  $\sqrt{\frac{\alpha}{a}} = \sqrt{2}$ ;  $\sqrt{\frac{\beta}{b}} = 2$ ;  $\sqrt{\frac{\alpha}{c}} = 2\sqrt{2}$ , one chooses  $C = \frac{1}{4}$ . Then such two mixed states are SLOCC equivalent.

Example 3. Let us consider another pair of mixed states in  $2 \otimes 2 \otimes 2$  system,

$$\rho_1 = \frac{1}{K} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\rho_2 = \frac{1}{2K} \begin{pmatrix} 1+b & 0 & 1-b & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & a+c & 0 & a-c & 0 & 0 & 0 & 0 \\ 1-b & 0 & 1+b & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & a-c & 0 & a+c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} + \frac{1}{a} & 0 & -\frac{1}{a} + \frac{1}{c} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{b} + 1 & 0 & -1 + \frac{1}{b} \\ 0 & 0 & 0 & 0 & -\frac{1}{a} + \frac{1}{c} & 0 & \frac{1}{c} + \frac{1}{a} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & -1 + \frac{1}{b} & 0 & 1 + \frac{1}{b} \end{pmatrix},$$

where the normalization factor  $K = \frac{3}{2} + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ .  $\rho_1$  and  $\rho_2$  have the same eigenvalues,  $\Lambda_1 = \Lambda_2 = \frac{1}{K} \text{diag}(\frac{1}{c}, \frac{1}{b}, \frac{1}{a}, \frac{3}{2}, a, b, c, \frac{1}{2})$ . Now we consider the case with different  $a$ ,  $b$ , and  $c$  unequal to 0, 1,  $\frac{2}{3}$ ,  $\frac{3}{2}$ , 2,  $\frac{1}{2}$ , which implies that  $\rho_1$  and  $\rho_2$  are not degenerated. In such case,

$$X = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$Y^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Let  $B$  be the identity matrix. Then

$$XBY^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that  $\text{rank}(R(XBY^\dagger)_{1|23}) = 1$ . Furthermore  $XBY = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Hence all nondegenerated mixed states  $\rho_1$  and  $\rho_2$  are SLOCC equivalent.

Now, we give one example for two quantum states non SLOCC equivalence. In fact, there are too many examples for two quantum states non SLOCC equivalence.

Example 4. Suppose  $|\psi\rangle_1 = \frac{1}{\sqrt{2}}(|001\rangle) + |010\rangle$ ,  $|\psi\rangle_2 = \frac{1}{\sqrt{2}}(|101\rangle) + |011\rangle$ .

On one hand, the coefficient matrices of these two pure states have different ranks, by Theorem 2, we can easily to determine that they are non SLOCC equivalence. On the other hand, we can also check their non SLOCC equivalence by Theorem 3. Since these are pure states and their density matrices is rank one, therefore their density matrices have only one nonzero eigenvalue 1. In this case, we can choose  $B$  as identity matrix, the  $X$  and  $Y$  can easily respectively obtained. One has

$$XBY^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that  $\text{rank}(R(XBY)_{1|23}) \neq 1$ . By Theorem 3, they are non SLOCC equivalence.

## V. CONCLUSIONS AND REMARKS

We have studied the SLOCC equivalence for arbitrary dimensional multipartite quantum states. Utilizing coefficient matrix and realignment, we present necessary and sufficient criteria for multipartite pure states and mixed states respectively. These conditions can be used to classify some SLOCC equivalent quantum states having the same rank. Some detailed examples are given to identify the SLOCC equivalence or non SLOCC equivalence. However, our methods have to recognize its disadvantage in determining the SLOCC equivalence for degenerate state. The reason is that the normalized eigenvectors of degenerate states can not be determined up to some unitary matrix. Thus the choose of unitary matrices  $X$  and  $Y$  in Eq.(10) can not be determined up to some unknown unitary matrices, which takes infinite possibility. Therefore, to check Eq.(10) becomes terribly difficult since one should check all possible choices.

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